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Rate of Convergence for the Absolutely (C, α) Summable Fourier Series of Functions of Bounded Variation

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A theorem of Bosanquet states that the Fourier series of a 2π-periodic function of bounded variation is absolutely (C, α) summable. In this paper we give a quantitative version of Bosanquet's result. © 1999 Academic Press

1. INTRODUCTION

Let f be a 2π-periodic and Lebesgue integrable function on $[-\pi, \pi]$. With each such function f we associate a Fourier series

$$S(f, x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt, \quad k = 0, 1, ...$$

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If f is a function of bounded variation on $[-\pi, \pi]$, the well-known theorem of Dirichlet–Jordan [5, p. 57] states that

$$\lim_{n\to\infty} S_n(f, x) = \tfrac{1}{2}(f(x+0)+f(x-0)),$$

where $(S_n(f))$ is the sequence of partial sum of the Fourier series of f.

A quantitative version of Dirichlet–Jordan's theorem was given in [1], where it was shown that, for $n \geqslant 1$,

$$\left| S_n(f, x) - \frac{1}{2}(f(x+0)+f(x-0)) \right| \leqslant \frac{3}{n} \sum_{k=1}^{n} Var_0^{\pi/k}(\varphi_x),$$

where

$$\varphi_x(t) = \begin{cases} f(x+t)+f(x-t)-(f(x+0)+f(x-0)), & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{1.1}$$

and $Var_a^b(g)$ is the total variation of g on $[a, b]$.

This result was obtained by using the integral representation of the sequence of partial sums

$$S_n(f, x) = \frac{1}{2}(f(x+0)+f(x-0)) + \frac{1}{2\pi} \int_0^\pi \varphi_x(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\, dt.$$

In this paper we want to establish a similar result for the sequence of (C, α), $\alpha > 0$, means of the partial sums $(S_n(f))$ by a different method. The (C, α) means of the sequence $(S_n(f))$ are defined by

$$S_n^\alpha(f) = \frac{1}{B_n^\alpha} \sum_{k=0}^{n} B_{n-k}^{\alpha-1} S_k(f), \qquad B_n^\alpha = \binom{n+\alpha}{n}.$$

If f is a 2π-periodic function, integrable on $[-\pi, \pi]$ and such that the limits $f(x+0)$, $f(x-0)$ exist, by a theorem of M. Riesz [5, p. 94], we have:

$$\lim_{n\to\infty} S_n^\alpha(f, x) = \tfrac{1}{2}(f(x+0)+f(x-0)).$$

A quantitative version of this result for Fourier Stieltjes series was obtained by S. M. Mazhar [4]. Mazhar's result is based on the estimates for the kernel in the integral representation of the sequence of (C, α) means. The method used in this paper is based on the concept of absolute convergence.

L. S. Bosanquet [3] has proved that the sequence $(S_n^\alpha(f))$ is absolutely convergent for every function f of bounded variation on $[-\pi, \pi]$ and every $\alpha > 0$. This means that

$$\sum_{k=1}^{\infty} |S_k^\alpha(f) - S_{k-1}^\alpha(f)| < \infty.$$

Using the terminology of summability theory, we can restate Bosanquet's result by saying that the Fourier series of a function of bounded variation is summable $|C, \alpha|$ for $\alpha > 0$. From these two results it follows immediately that

$$|S_n^\alpha(f, x) - \tfrac{1}{2}(f(x+0) + f(x-0))| \leqslant \sum_{k=n+1}^{\infty} |S_k^\alpha(f, x) - S_{k-1}^\alpha(f, x)| \stackrel{\text{def}}{=} R_n^\alpha(f, x). \tag{1.2}$$

In this paper we will obtain the following estimates for $R_n^\alpha(f, x)$.

THEOREM. *Let f be a 2π-periodic function of bounded variation on $[-\pi, \pi]$. Then, for $\alpha > 0$ and $n \geqslant 2$ we have*

$$R_n^\alpha(f, x) \leqslant \frac{4\alpha}{n\pi} \sum_{k=1}^{n} Var_0^{\pi/k}(\varphi_x), \qquad x \in \mathbb{R}. \tag{1.3}$$

Since φ_x is a function of bounded variation, continuous at $t = 0$, we have $Var_0^{\pi/n}(\varphi_x) \to 0$ as $n \to \infty$. Consequently, the right hand side of (1.3) tends to 0 as $n \to \infty$.

In Section 2 we will give the proof of inequality (1.3).

A result related to this theorem for $-1 < \alpha \leqslant 0$ was proved in [2].

2. $|C, \alpha|$ SUMMABILITY

The proof of our main result is based on the following Lemmas.

LEMMA 2.1. *For $\alpha > 0$ let*

$$\tau_n^\alpha(f, x) = \sum_{k=1}^{n} \frac{B_{n-k}^{\alpha-1}}{B_n^\alpha} k(a_k \cos kx + b_k \sin kx). \tag{2.1}$$

Then, for $n \geqslant 1$ we have:

$$\tau_n^\alpha(f, x) = n(S_n^\alpha(f, x) - S_{n-1}^\alpha(f, x)). \tag{2.2}$$

Remark. For $\alpha = 1$ the relation (2.2) reduces to the identity

$$\frac{1}{n+1} \sum_{k=1}^{n} k(a_k \cos kx + b_k \sin kx) = n(\sigma_n(f, x) - \sigma_{n-1}(f, x)),$$

which is better known in the form

$$\frac{1}{n+1} \sum_{k=1}^{n} k(a_k \cos kx + b_k \sin kx) = S_n(f, x) - \sigma_n(f, x).$$

Proof of Lemma 1. We have

$$\begin{aligned}\sum_{k=1}^{n} & B_{n-k}^{\alpha-1} k(S_k(f) - S_{k-1}(f)) \\ &= \sum_{k=0}^{n} B_{n-k}^{\alpha-1} k S_k(f) - \sum_{k=0}^{n-1} B_{n-1-k}^{\alpha-1} (k+1)\, S_k(f) \\ &= nB_n^\alpha S_n^\alpha(f) - \sum_{k=0}^{n-1} (B_{n-k}^{\alpha-1}(n-k) + B_{n-1-k}^{\alpha-1}(k+1))\, S_k(f).\end{aligned}$$

Since

$$(n-k)\, B_{n-k}^{\alpha-1} = (\alpha + n - 1 - k)\, B_{n-1-k}^{\alpha-1},$$

it follows that

$$\sum_{k=0}^{n-1} (B_{n-k}^{\alpha-1}(n-k) + B_{n-1-k}^{\alpha-1}(k+1))\, S_k(f) = (n+\alpha) \sum_{k=0}^{n-1} B_{n-1-k}^{\alpha-1} S_k(f).$$

Hence,

$$\sum_{k=1}^{n} B_{n-k}^{\alpha-1} k(S_k(f) - S_{k-1}(f)) = nB_n^\alpha S_n^\alpha(f) - (n+\alpha)\, B_{n-1}^{\alpha-1} S_{n-1}^\alpha(f)$$

and Lemma 1 has been proved since $(n+\alpha)\, B_{n-1}^{\alpha-1} = nB_n^\alpha$. ∎

Lemma 2 provides key estimates for the (C, α) means

$$S_n^\alpha(x) = \sum_{k=1}^{n} \frac{B_{n-k}^{\alpha-1}}{B_n^\alpha} \sin kx$$

of the sequence $(\sin kx)$.

LEMMA 2.2. *Let* $\alpha > 0$. *Then*

$$|S_n^\alpha(x)| \leqslant n\,|x|, \qquad x \in \mathbb{R}, \quad n = 1, 2, ..., \tag{2.3}$$

and

$$|S_n^\alpha(x)| \leqslant \frac{2\pi\alpha}{n\,|x|}, \qquad x \in \mathbb{R}, \quad n = 2, 3, ... \tag{2.4}$$

Proof of Lemma 2. Without loss of generality we may assume that $0 < x < \pi$. Since

$$2\sin\frac{x}{2}\sin kx = \cos\left(k - \frac{1}{2}\right)x - \cos\left(k + \frac{1}{2}\right)x,$$

we have

$$\begin{aligned} 2\sin\frac{x}{2} & \sum_{k=1}^{n} B_{n-k}^{\alpha-1}\sin kx \\ &= \sum_{k=1}^{n} B_{n-k}^{\alpha-1}\cos\left(k-\frac{1}{2}\right)x - \sum_{k=1}^{n} B_{n-k}^{\alpha-1}\cos\left(k+\frac{1}{2}\right)x \\ &= B_{n-1}^{\alpha-1}\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x + \sum_{k=1}^{n-1}(B_{n-1-k}^{\alpha-1} - B_{n-k}^{\alpha-1})\cos\left(k+\frac{1}{2}\right)x. \end{aligned}$$

Hence,

$$2\sin\frac{x}{2}\left|\sum_{k=1}^{n} B_{n-k}^{\alpha-1}\sin kx\right| \leqslant B_{n-1}^{\alpha-1} + 1 + \sum_{k=1}^{n-1}|B_{n-1-k}^{\alpha-1} - B_{n-k}^{\alpha-1}|.$$

Since $B_m^\alpha - B_{m-1}^\alpha = B_m^{\alpha-1}$, we have

$$\begin{aligned} 2\sin\frac{x}{2}\left|\sum_{k=1}^{n} B_{n-k}^{\alpha-1}\sin kx\right| &\leqslant B_{n-1}^{\alpha-1} + 1 + \sum_{k=1}^{n-1} B_{n-k}^{\alpha-2} \\ &\leqslant B_{n-1}^{\alpha-1} + 1 + \sum_{k=1}^{n-1} B_{k}^{\alpha-2} \\ &= B_{n-1}^{\alpha-1} + \sum_{k=0}^{n-1} B_{k}^{\alpha-2} = 2B_{n-1}^{\alpha-1}. \end{aligned}$$

Hence,

$$\left|\sum_{k=1}^{n}\frac{B_{n-k}^{\alpha-1}}{B_n^\alpha}\sin kx\right| \leqslant \frac{1}{\sin\dfrac{x}{2}}\,\frac{B_{n-1}^{\alpha-1}}{B_n^\alpha},$$

and Lemma 2 has been proved since

$$\frac{B_{n-1}^{\alpha-1}}{B_n^\alpha} = \frac{\alpha n}{(n+\alpha)(n+\alpha-1)} \leqslant \frac{\alpha}{n-1} \leqslant \frac{2\alpha}{n}$$

for $n \geqslant 2$, and $\sin(x/2) \geqslant (x/\pi)$ for $0 < x \leqslant \pi$. ∎

Proof of the Theorem. By Lemma 1 we have

$$R_n^\alpha(f) = \sum_{k=n+1}^{\infty} |S_k^\alpha(f) - S_{k-1}^\alpha(f)| = \sum_{k=n+1}^{\infty} \frac{|\tau_k^\alpha|}{k}. \tag{2.5}$$

Since

$$\begin{aligned} a_k \cos kx + b_k \sin kx &= \frac{1}{\pi} \int_0^\pi [f(x+t) + f(x-t)] \cos kt \, dt \\ &= \frac{1}{\pi} \int_0^\pi \varphi_x(t) \cos kt \, dt, \end{aligned}$$

we have by (2.1)

$$\tau_n^\alpha(f, x) = \frac{1}{\pi} \sum_{k=1}^{n} \frac{B_{n-k}^{\alpha-1}}{B_n^\alpha} k \int_0^\pi \varphi_x(t) \cos kt \, dt.$$

Since

$$k \int_0^\pi \varphi_x(t) \cos kt \, dt = \int_0^\pi \varphi_x(t) \, d \sin kt = -\int_0^\pi \sin kt \, d\varphi_x(t),$$

the preceding formula now becomes

$$\tau_n^\alpha(f, x) = -\frac{1}{\pi} \int_0^\pi \left(\sum_{v=1}^{k} \frac{B_{k-v}^{\alpha-1}}{B_k^\alpha} \sin vt \right) d\varphi_x(t),$$

that is

$$\tau_n^\alpha(f, x) = -\frac{1}{\pi} \int_0^\pi S_n^\alpha(t) \, d\varphi_x(t).$$

Since φ_x is a function of bounded variation, we have

$$|\tau_n^\alpha(f, x)| \leqslant \frac{1}{\pi} \int_0^\pi |S_n^\alpha(t)| \, dVar_0^t(\varphi_x).$$

Using this inequality and (2.5), we find that

$$R_n^\alpha(f, x) = \frac{1}{\pi} \int_0^\pi \sum_{k=n+1}^\infty \frac{|S_k^\alpha(t)|}{k} \, dVar_0^t(\varphi_x) = \mathscr{A}_n^\alpha + \mathscr{B}_n^\alpha, \tag{2.6}$$

where

$$\mathscr{A}_n^\alpha = \frac{1}{\pi} \int_0^{\pi/n} \sum_{k=n+1}^\infty \frac{|S_k^\alpha(t)|}{k} \, dVar_0^t(\varphi_x) \tag{2.7}$$

and

$$\mathscr{B}_n^\alpha = \frac{1}{\pi} \int_{\pi/n}^{\pi} \sum_{k=n+1}^\infty \frac{|S_k^\alpha(t)|}{k} \, dVar_0^t(\varphi_x). \tag{2.8}$$

We will estimate $\mathscr{A}_n^\alpha$ first. Let $0 < \delta < \pi/n$. We have

$$\begin{aligned} \int_\delta^{\pi/n} \sum_{k=n+1}^\infty \frac{|S_k^\alpha(t)|}{k} \, dVar_0^t(\varphi_x) &\leqslant \int_\delta^{\pi/n} \sum_{k=1}^\infty \frac{|S_k^\alpha(t)|}{k} \, dVar_0^t(\varphi_x) \\ &\leqslant \int_\delta^{\pi/n} \left(\sum_{k \leqslant \pi/t} + \sum_{k > \pi/t} \right) \frac{|S_k^\alpha(t)|}{k} \, dVar_0^t(\varphi_x). \end{aligned} \tag{2.9}$$

Using inequality (2.3), we find that

$$\sum_{k \leqslant \pi/t} \frac{|S_k^\alpha(t)|}{k} \leqslant \sum_{k \leqslant \pi/t} t \leqslant t \left[\frac{\pi}{t} \right].$$

Hence,

$$\sum_{k \leqslant \pi/t} \frac{|S_k^\alpha(t)|}{k} \leqslant \pi. \tag{2.10}$$

Using inequality (2.4), we find that

$$\sum_{k > \pi/t} \frac{|S_k^\alpha(t)|}{k} \leqslant 2\pi\alpha \sum_{k > \pi/t} \frac{1}{k^2 t} \leqslant \frac{2\pi\alpha}{t} \sum_{k=[\pi/t]+1}^\infty \frac{1}{k^2} \leqslant \frac{2\pi\alpha}{t} \frac{1}{\left[\frac{\pi}{t} \right] + 1}.$$

Hence,

$$\sum_{k > \pi/t} \frac{|S_k^\alpha(t)|}{k} \leqslant 2\alpha. \tag{2.11}$$

Using (2.9) and the inequalities (2.10) and (2.11), we find that

$$\int_{\delta}^{\pi/n} \sum_{k=1}^{\infty} \frac{|S_k^{\alpha}(t)|}{k} \, dVar_0^t(\varphi_x) \leqslant \int_0^{\pi/n} (\pi + 2\alpha) \, dVar_0^t(\varphi_x)$$

$$\leqslant (\pi + 2\alpha) \, Var_0^{\pi/n}(\varphi_x).$$

Since this inequality is valid for any δ, it follows that

$$\mathscr{A}_n^{\alpha} = \frac{1}{\pi} \int_0^{\pi/n} \sum_{k=1}^{\infty} \frac{|S_k^{\alpha}(t)|}{k} \, dVar_0^t(\varphi_x) \leqslant \left(1 + \frac{2\alpha}{\pi^{\alpha+1}}\right) Var_0^{\pi/n}(\varphi_x). \tag{2.12}$$

Next we estimate $\mathscr{B}_n^{\alpha}$. From the inequality (2.4) it follows that

$$\mathscr{B}_n^{\alpha} \leqslant 2\alpha \int_{\pi/n}^{\pi} \frac{1}{t} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \, dVar_0^t(\varphi_x) \leqslant \frac{2\alpha}{n} \int_{\pi/n}^{\pi} \frac{1}{t} \, dVar_0^t(\varphi_x).$$

Integrating by parts, we find that

$$\int_{\pi/n}^{\pi} \frac{1}{t} \, dVar_0^t(\varphi_x) = \frac{1}{\pi} Var_0^{\pi}(\varphi_x) - \frac{n}{\pi} Var_0^{\pi/n}(\varphi_x) + \int_{\pi/n}^{\pi} \frac{Var_0^t(\varphi_x)}{t^2} \, dt.$$

Hence,

$$\mathscr{B}_n^{\alpha} \leqslant \frac{2\alpha}{n\pi} Var_0^{\pi}(\varphi_x) - \frac{2\alpha}{\pi} Var_0^{\pi/n}(\varphi_x) + \frac{2\alpha}{n} \int_{\pi/n}^{\pi} \frac{Var_0^t(\varphi_x)}{t^2} \, dt. \tag{2.13}$$

Using inequalities (2.6), (2.12), and (2.13), we find that

$$R_n^{\alpha}(f, x) \leqslant \frac{2\alpha}{n\pi} Var_0^{\pi}(\varphi_x) + \frac{2\alpha}{n} \int_{\pi/n}^{\pi} \frac{Var_0^t(\varphi_x)}{t^2} \, dt.$$

Replacing t by π/t in the last integral, we obtain

$$\int_{\pi/n}^{\pi} \frac{Var_0^t(\varphi_x)}{t^2} \, dt = \frac{1}{\pi} \int_1^n Var_0^{\pi/t}(\varphi_x) \, dt \leqslant \frac{1}{\pi} \sum_{k=1}^{n} Var_0^{\pi/k}(\varphi_x).$$

Hence,

$$R_n^{\alpha}(f, x) \leqslant \frac{2\alpha}{n\pi} \left(Var_0^{\pi}(\varphi_x) + \sum_{k=1}^{n} Var_0^{\pi/k}(\varphi_x) \right),$$

that is

$$R_n^{\alpha}(f, x) \leqslant \frac{4\alpha}{n\pi} \sum_{k=1}^{n} Var_0^{\pi/k}(\varphi_x)$$

and, thus, the Theorem has been proved. ∎

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